University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO PROBLEM SET 7

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Reading:
105 Notes 8.1-8.3, 6.1-6.2 (again).
Hand & Finch 4.7, 5.1.

1.
[This a (hopefully clearer) version of Hand & Finch 4.17, "tetherball".] A mass m is attached to a weightless string that initially has a length s_0. The other end of the string is attached to a post of radius a. Neglect the effect of gravity. Suppose that the mass is set into motion. It is given an initial velocity of magnitude v_0 directed so that the string remains taut. The string wraps itself around the post, causing the mass to spiral inward toward it.

(**a**)
Write the Lagrangian in terms of $\dot{x}$ and $\dot{y}$, the cartesian velocity components of the mass. Is there a potential energy term?

Solution:

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right)$$

There is no potential energy term.

(**b**)
Use as generalized coordinates $s(t)$, the length of the part of the string that is not yet in contact with the post, and $\psi(t)$, the azimuthal angle at which the string barely fails to make contact with the post. Express $\dot{x}$ and $\dot{y}$ in terms of these generalized coordinates and their time derivatives.

Solution:
Letting the origin be at the center of the post, and letting ψ be the counter-clockwise angle from the x axis, we can write x and y as:

$$\begin{aligned} x &= a\cos\psi - s\sin\psi \\ y &= a\sin\psi + s\cos\psi \end{aligned}$$

Taking the time derivative of these expressions yields:

$$\begin{aligned} \dot{x} &= -a\dot{\psi}\sin\psi - \dot{s}\sin\psi - s\dot{\psi}\cos\psi \\ \dot{y} &= +a\dot{\psi}\cos\psi + \dot{s}\cos\psi - s\dot{\psi}\sin\psi \end{aligned}$$

(**c**)
Write a (constraint) equation relating $\dot{s}$ to $\dot{\psi}$. Use it to greatly simplify your answers for (**b**). Rewrite the Lagrangian using s as the only generalized coordinate.

Solution:
Since the string is winding up on the post, and thus decreasing its length, we must have:

$$\dot{s} = -a\dot{\psi}$$

Plugging this into our expressions for $\dot{x}$ and $\dot{y}$ gives us:

$$\begin{aligned} \dot{x} &= \frac{s\dot{s}}{a}\cos\psi \\ \dot{y} &= \frac{s\dot{s}}{a}\sin\psi \end{aligned}$$

And so the lagrangian becomes:

$$\mathcal{L} = \frac{m}{2a^2}s^2\dot{s}^2$$

(**d**)
Use the Euler-Lagrange equation to obtain an equation of motion for s. (You don't need to solve it.)

Solution:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{s}}\right) &= \frac{\partial\mathcal{L}}{\partial s} \\ \frac{d}{dt}\left(\frac{m}{a^2}s^2\dot{s}\right) &= \frac{m}{a^2}s\dot{s}^2 \\ s\ddot{s} + \dot{s}^2 &= 0 \end{aligned}$$

(e)
Since the Lagrangian has no explicit time dependence, and it depends quadratically on $\dot{s}$, the total energy is conserved. Write an equation setting the initial energy (expressed in terms of v_0) equal to the energy at an arbitrary value of s (expressed in terms of s and $\dot{s}$).
Solution:

$$E = \mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{s}}\dot{s} - \mathcal{L}$$
$$= \frac{m}{2a^2}s^2\dot{s}^2 = \frac{m}{2}v_o^2$$

(f)
Use this equation to express dt in terms of ds multiplied by a function of s. Integrate it to solve for the time T that elapses before the mass hits the post. You should obtain the simple result

$$T = \frac{s_0^2}{2av_0} \ .$$

Solution:

$$\dot{s}^2 = \frac{a^2v_o^2}{s^2}$$
$$\dot{s} = -\frac{av_o}{s}$$
$$dt = -\frac{s}{av_o}ds$$
$$T = -\int_{s_o}^{0} \frac{s}{av_o}ds$$
$$= \frac{1}{av_o}\int_0^{s_o} s\ ds$$
$$= \frac{s_o^2}{2av_o}$$

Note that this could also be found by realizing that the DE in part (d) can also be written as $\frac{d}{dt}(s\dot{s}) = 0$. This, along with the initial conditions that $s(0) = s_o$ and $\dot{s}(0) = -\frac{av_o}{s_o}$ (obtainable from the energy expression), allows us to show that:

$$s(t) = \sqrt{s_o^2 - 2av_ot}$$

(g)
Is the angular momentum of the mass about the *axis* of the post conserved in this problem? Why or why not?
Solution:
Since the mass is constrained to lie in the xy plane, there is only a z component of angular momentum:

$$L_z = m(x\dot{y} - y\dot{x})$$
$$= -\frac{ms^2\dot{s}}{a}$$

whose magnitude is diminishing with time. Angular momentum is not conserved because the string is not directed toward the center of the post. Hence the tension in the string exerts a *torque* on the mass with respect to the post's center.

2.
Hand & Finch 4.19.
Solution:
(a)

$$\tau = 2\pi\sqrt{\frac{\mu}{k}}a^{\frac{3}{2}} \ \text{(Hand \& Finch Eq. 4.61)}\ ,$$

where $k = GM_sm_e$, $\mu \approx m_e$, and $a = R_e$. Also, $\tau = 1$ year $= 3.15 \times 10^7$ s. This allows us to solve for M_s:

$$M_s = \frac{4\pi^2{R_e}^3}{G\tau^2}$$
$$= 1.97 \times 10^{30}\ \text{kg}$$

(b)

$$m_e = \frac{4\pi^2{R_m}^3}{G{\tau_m}^2}$$
$$\frac{M_s}{m_e} = \frac{{R_e}^3}{{\tau_e}^2}\frac{{\tau_m}^2}{{R_m}^3}$$
$$= 3.38 \times 10^5$$
$$m_e = M_s\frac{m_e}{M_s}$$
$$= 5.83 \times 10^{24}\ \text{kg}$$

This is close to the actual value of 5.98×10^{24} kg. The density of the earth is

$$\rho_e = \frac{m_e}{\frac{4\pi}{3}{r_e}^3}$$
$$= 5.36 \times 10^3\ \frac{\text{kg}}{\text{m}^3} = 5.36\ \frac{\text{g}}{\text{cm}^3}$$

The mass of the moon (1.23% of the earth's mass) could be similarly determined by observing the effects of its gravity on other objects, such as spacecraft, orbiting around it. Also, one could compare the height of the tides when the sun is aligned *vs.* anti-aligned with the moon: knowing the radii of the moon's and earth's orbits from their periods, one can solve for the ratio of the moon's and sun's masses (see problem 3). Finally, the moon's mass may be measured from its perturbations on the earth's orbit – but this is a dense topic.

3.
Hand & Finch 4.21.
Solution:
(**a**)
To quote Feynman: "The pull of the Moon for the Earth and for the water is 'balanced' at the center. But the water which is closer to the Moon is pulled *more* than the average and the water which is farther away from it is pulled *less* than the average. Furthermore, the water can flow while the more rigid Earth cannot..."(Feynman Lectures I). So the near water gets pulled away from the Earth, which in turn gets pulled away from the far water. This causes a net 'elongation' of the Earth and its oceans, directed approximately along the line joining the Earth and Moon. There is one high tide at the front of the Earth and one at the back.
(**b**)
Since the Earth is rotating, the direction of the elongation of the Earth is always changing with respect to the Earth's surface. The dual bulges of water on either side of the Earth cannot change their position instantly (because of viscous friction and the water's inertia) so there is a constant phase lag between the direction of elongation and the Earth-Moon direction. A similar tidal effect can be used to explain why only one side of the Moon ever faces the Earth.
(**c**)

$$\begin{aligned}
F_{\text{tide, sun}} &= F_{s-e}(R_e - r_e) - F_{s-e}(R_e + r_e) \\
&\approx 2r_e \frac{\partial F_{s-e}}{\partial r}\Big|_{R_e} \\
&= \frac{4r_e G M_s m_e}{{R_e}^3} \\
F_{\text{tide, moon}} &\approx 2r_e \frac{\partial F_{m-e}}{\partial r}\Big|_{R_m} \\
&= \frac{4r_e G m_m m_e}{{R_m}^3} \\
\frac{F_{\text{tide, sun}}}{F_{\text{tide, moon}}} &= \frac{M_s}{{R_e}^3}\frac{{R_m}^3}{m_m} \\
&= 0.45
\end{aligned}$$

The two tidal forces are of the same order of magnitude.

4.
Consider a particle of mass m that is constrained to move on the surface of a paraboloid whose equation (in cylindical coordinates) is $r^2 = 4az$. If the particle is subject to a gravitational force $-mg\hat{\mathbf{z}}$, show that the frequency of small oscillations about a circular orbit with radius $\rho = \sqrt{4az_0}$ is

$$\omega = \sqrt{\frac{2g}{a + z_0.}}$$

Solution:
In cylindrical coordinates (r, θ, z), we write the Lagrangian as

$$\mathcal{L} = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 + \tfrac{1}{2}m\dot{z}^2 - mgz \ .$$

Use the equation of constraint $r^2 = 4az$ to get rid of z:

$$\mathcal{L} = \tfrac{1}{2}m\dot{r}^2 + \frac{mr^2\dot{r}^2}{8a^2} + \tfrac{1}{2}mr^2\dot{\theta}^2 - \frac{mgr^2}{4a} \ .$$

The Euler-Lagrange equation for θ just expresses angular momentum conservation: $l \equiv mr^2\dot{\theta}$ is constant. The equation for r is

$$\ddot{r}\left(1 + \frac{r^2}{4a^2}\right) + \frac{r\dot{r}^2}{4a^2} = \frac{l^2}{m^2r^3} - \frac{gr}{2a}, \qquad (1)$$

where we have substituted $\dot{\theta} = l/mr^2$ and canceled a factor m. Choose ρ such that $r(t) = \rho$

is the constant-radius solution to this equation, and consider small perturbations about this solution: $r(t) = \rho + x(t)$, with $x \ll \rho$. Drop all terms with more than one power of x in our differential equation for r, and you find that

$$\ddot{x}\left(1+\frac{\rho^2}{4a^2}\right) = \frac{l^2}{m^2(\rho+x)^3} - \frac{g(\rho+x)}{2a}$$
$$= \frac{l^2}{m^2\rho^3}\left(1-\frac{3x}{\rho}\right) - \frac{g}{2a}(\rho+x)\ .$$

(The last step comes from a Taylor series expansion of $(\rho+x)^{-3}$.) Since $r(t) = \rho$ is a solution to equation (1), we know that $l^2/m^2\rho^3 = g\rho/2a$. (This comes from setting $\dot{r} = \ddot{r} = 0$ in equation (1).) Use this to simplify the differential equation for x:

$$\ddot{x}\left(1+\frac{\rho^2}{4a^2}\right) + \frac{2g}{a}x = 0$$

This is the equation for a harmonic oscillator with frequency

$$\omega = \sqrt{\frac{2g/a}{1+\rho^2/4a^2}} = \sqrt{\frac{2g}{a+z_0}}.$$

5.

An orbit that is almost circular can be considered to be a circular orbit to which a small perturbation has been applied. Take ρ to be the (unperturbed) circular orbit radius and define

$$g(r) = \frac{1}{\mu}\frac{\partial U(r)}{\partial r}\ ,$$

where μ is the reduced mass and U is an arbitrary potential. Set the radius $r = \rho + x$, where x is a small perturbation.

(**a**)

Starting from the differential equation for r and using the fact that the angular momentum l is constant, substitute $r = \rho + x$. Retaining terms only to first order in x, Taylor expand $g(r)$ about the point $r = \rho$, and show that x satisfies the differential equation

$$\ddot{x} + \left[\frac{3g(\rho)}{\rho} + g'(\rho)\right]x = 0\ ,$$

where $g'(\rho)$ is dg/dr evaluated at $r = \rho$.

Solution:

The Lagrangian for a particle moving in a central force field is

$$\mathcal{L} = \tfrac{1}{2}\mu\dot{r}^2 + \tfrac{1}{2}\mu r^2\dot{\theta}^2 - U(r)$$

where r and θ are polar coordinates. The Euler-Lagrange equation for θ says that $l = \mu r^2\dot{\theta}$ is constant, and the equation for r is

$$\mu\ddot{r} = \mu r\dot{\theta}^2 - U' = \frac{l^2}{\mu r^3} - \mu g(r)$$

Let ρ be the radius of the constant-r solution to this equation. Then ρ satisfies $l^2/\mu^2\rho^3 = g(\rho)$. Now we can look for solutions of the form $r(t) = \rho + x(t)$, with $x \ll \rho$. Dropping all terms of higher than first order in x, our differential equation becomes

$$\ddot{x} = \frac{l^2}{\mu^2\rho^3}\left(1-\frac{3x}{\rho}\right) - g(\rho) - xg'(\rho)$$

where we have done a Taylor expansion in x of $(\rho+x)^{-3}$ and $g(\rho+x)$. Now use our equation for ρ above to simplify this:

$$\ddot{x} = g(\rho)\left(1-\frac{3x}{\rho}\right) - g(\rho) - xg'(\rho)$$
$$= -\left(\frac{3g(\rho)}{\rho} + g'(\rho)\right)x$$

(**b**)

Taking the force law to be $F(r) = -kr^{-n}$, where n is an integer, show that the angle between two successive values of $r = r_{\max}$ (the "apsidal angle") is $2\pi/\sqrt{3-n}$. Thus, if $n > -6$, show that in general a closed orbit will result only for the harmonic oscillator force and the inverse square law force.

Solution:

$g(r) = \frac{k}{\mu}r^{-n}$, so $g'(r) = -\frac{k}{\mu}nr^{-(n+1)}$. Plug that into our equation for $\ddot{x}$ and you'll find that

$$\ddot{x} + \left(\frac{k}{\mu}\rho^{-(n+1)}(3-n)\right)x = 0.$$

This is the equation for a harmonic oscillator with frequency

$$\omega_x = \left(\frac{k}{\mu}\rho^{-(n+1)}(3-n)\right)^{1/2}$$

We need to compare this to ω_θ, the angular frequency of the unperturbed circular motion. Use the centripetal force equation $F_{\text{cent}} = \mu\rho\dot{\theta}^2$ to get

$$\omega_\theta = \dot{\theta} = \left(\frac{U'}{\mu\rho}\right)^{1/2} = \left(\frac{k}{\mu}\rho^{-(n+1)}\right)^{1/2}$$

Successive maxima of r occur when $\omega_x t$ increases by 2π, and the angle through which the particle has moved in that time is

$$\Delta\theta = \omega_\theta t = 2\pi\frac{\omega_\theta}{\omega_x} = \frac{2\pi}{\sqrt{3-n}}$$

If $\Delta\theta/2\pi$ is a rational number, say j/k with j and k integers, then after j orbits the path will close. If $\Delta\theta$ is irrational, the orbit will never close. Looking at our expression for $\Delta\theta$, it's clear that the orbit closes only if $3-n$ is a perfect square, which only happens for $n > -6$ if $n = 2$ or $n = -1$.

6.
Consider the motion of a particle in a central force field $F = -k/r^2 + C/r^3$.
(**a**)
Show that the equation of the orbit can be put in the form

$$\frac{1}{r} = \frac{1+\epsilon\cos\alpha\theta}{a(1-\epsilon^2)}\ ,$$

which is an ellipse for $\alpha = 1$, but is a *precessing* ellipse for $\alpha \neq 1$.
Solution:
Start from Eq. (7.8) in the lecture notes:

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu k}{l^2}$$

Multiply it by u^2l^2/μ. Then, in place of the gravitational force $-ku^2$, substitute the full force $-ku^2 + Cu^3$. This yields

$$\frac{l^2u^2}{m}(u''+u) = ku^2 - Cu^3\ ,$$

where u'' denotes $d^2u/d\theta^2$. Rearranging this equation, we get

$$u'' + \left(1+\frac{mC}{l^2}\right)u - \frac{km}{l^2} = 0$$

This looks like the differential equation for a harmonic oscillator, plus a constant displacement. So we know it has a solution of the form $u = A + B\cos\alpha\theta$. Substituting this into the differential equation, we find that we get a valid solution as long as

$$\alpha = \sqrt{1+\frac{mC}{l^2}} \text{ and } A = \frac{km/l^2}{1+mC/l^2}$$

(B is arbitrary). Comparing this with the form given for a precessing ellipse:

$$u = \frac{1}{r} = \frac{1+\epsilon\cos\alpha\theta}{a(1-\epsilon^2)}\ ,$$

we find that $a(1-\epsilon^2) = 1/A$ and $a(1-\epsilon^2)/\epsilon = 1/B$. If you wanted to, you could solve these for a and ϵ, but there's no need to.
(**b**)
The precessing motion may be described in terms of the *rate of precession of the perihelion*, where the term perihelion is used (loosely) to denote *any* of the turning points of the orbit. Derive an approximate expression for the rate of precession when α is close to unity, in terms of the dimensionless quantity $\eta = C/ka$.
Solution:
How fast is the ellipse precessing? Well, between successive maxima of r, θ increases by $2\pi/\alpha$, and if the ellipse weren't precessing at all, that angle would be 2π. So the amount of precession per revolution is $\Delta\theta = 2\pi(1-1/\alpha)$. Now let's assume α is close to 1. That means that $mC/l^2 \ll 1$. In this approximation, we can expand $1/\alpha = (1+mC/l^2)^{-1/2} \approx 1 - mC/2l^2$. (This is just a Taylor expansion.) So the precession rate is

$$\Delta\theta = \frac{\pi mC}{l^2}$$

per orbit. To write this in terms of η, just note from above that

$$a(1-\epsilon^2) = \frac{l^2}{km}(1 + mC/l^2) \approx l^2/km \ .$$

(Why were we able to drop the mC/l^2? Because it is small compared to 1.) Rearranging this, we get

$$\frac{m}{l^2} = \frac{1}{ka(1-\epsilon^2)} \ ,$$

and plugging that into our expression for $\Delta\theta$, we get

$$\Delta\theta = \frac{\pi C}{ka(1-\epsilon^2)} = \frac{\pi\eta}{(1-\epsilon^2)} \ .$$

(**c**)
The ratio η is a measure of the strength of the perturbing inverse cube term relative to the main inverse square term of the force. Show that the rate of precession of Mercury's perihelion ($40''$ of arc per century) could be accounted for *classically*, if $\eta = 1.42 \times 10^{-7}$. [Mercury's period and eccentricity are 0.24 y and 0.206, respectively.]
Solution:
If $\eta = 1.42 \times 10^{-7}$ and $\epsilon = 0.206$, then $\Delta\theta = 4.66 \times 10^{-7}$ radians $= 9.61'' \times 10^{-2}$. That's the precession per orbit, so to get the precession per century, we need to multiply by the number of orbits per century, 100/0.24. Then we find that $\Delta\theta = 40''$/century.

7.
A He nucleus with velocity $v = 0.05c$ is normally incident on an Au foil that is 1 micron (1×10^{-6} m) thick. What is the probability that it will scatter into the backward hemisphere, *i.e.* bounce off the foil? (Please supply a *number*.)
Solution:
First, what is the cross section that will result in the scattering of a He nucleus into the back hemisphere from a *single* Au nucleus?

$$\begin{aligned}
\sigma_{back} &= \int_{\theta=\frac{\pi}{2}}^{\theta=\pi} \frac{d\sigma}{d\Omega} d\Omega \\
&= \left(\frac{Zze^2}{2\mu v_o^2}\right)^2 \int_{\frac{\pi}{2}}^{\pi} \frac{2\pi \sin\theta \, d\theta}{\sin^4\frac{\theta}{2}} \\
&= 4\pi \left(\frac{Zze^2}{2\mu v_o^2}\right)^2 \int_{\frac{\pi}{2}}^{\pi} \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2} \, d\theta}{\sin^4\frac{\theta}{2}} \\
&= 8\pi \left(\frac{Zze^2}{2\mu v_o^2}\right)^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos u \, du}{\sin^3 u} \quad (u = \frac{\theta}{2}) \\
&= \pi \left(\frac{Zze^2}{\mu v_o^2}\right)^2
\end{aligned}$$

We also need the number of Au nuclei per unit area that the He nucleus sees in the foil. With a little thought, it can be seen that

$$\frac{\text{number}}{\text{unit Area}} = \frac{\rho_{\text{Au}} t}{m_{\text{Au}}} \ ,$$

where t is the foil thickness. The probability of backwards scattering from the foil is

$$\begin{aligned}
P_{\text{back}} &= \frac{\sigma_{\text{back}}}{\text{Area}_{\text{foil}}} \times (\#\text{ of Au nuclei in foil}) \\
&= \sigma_{\text{back}} \times \frac{\text{number}}{\text{unit Area}} \\
&= \frac{\pi \rho_{\text{Au}} t}{m_{\text{Au}}} \left(\frac{Zze^2}{\mu v_o^2}\right)^2
\end{aligned}$$

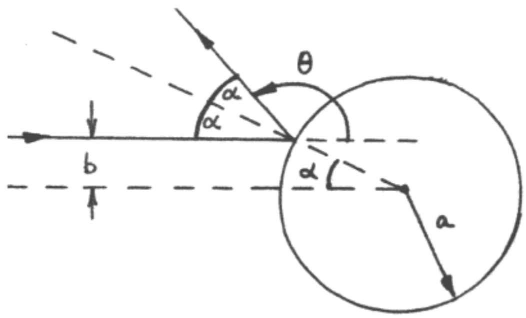
The numerical values (note we are working in cgs) of all these quantities are:

$$\begin{aligned}
\rho_{\text{Au}} &= 19.28 \ \frac{\text{g}}{\text{cm}^3} \\
t &= 10^{-4} \text{ cm} \\
m_{\text{Au}} &= 3.28 \times 10^{-22} \text{ g} \\
Z &= 79 \\
z &= 2 \\
e &= 4.8 \times 10^{-10} \text{ esu} \\
\mu &\approx m_{\text{He}} = 6.67 \times 10^{-24} \text{ g} \\
v_o &= .05 \ c = 1.5 \times 10^9 \ \frac{\text{cm}}{\text{s}}
\end{aligned}$$

Plugging in these values yields $P_{\text{back}} = 0.00011$, so most of the He nuclei do pass through the foil.

8.

Calculate the differential cross section $d\sigma/d\Omega$ and the total cross section σ_T for the elastic scattering of a point particle from an impenetrable sphere; *i.e.* the potential is given by $U(r) = 0,\ r > a$; $U(r) = \infty,\ r < a$.



Solution:

Consider a single particle approaching with impact parameter b and bouncing off of the sphere. Let θ be the scattering angle and let α be the angle the particle's trajectory makes with the normal to the sphere. The angle of reflection off the sphere equals the angle of incidence α. Therefore $2\alpha + \theta = \pi$. Also, the angle θ is related to b by

$$\begin{aligned} b &= a \sin\alpha \\ &= a \sin\frac{\pi - \theta}{2} \\ &= a\cos\frac{\theta}{2} \end{aligned}$$

The differential cross section is given by

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{b(\theta)}{\sin\theta}\left|\frac{db}{d\theta}\right| \\ &= \frac{a\cos\frac{\theta}{2}}{\sin\theta}\left|-\frac{a}{2}\sin\frac{\theta}{2}\right| \\ &= \frac{a^2}{4} \end{aligned}$$

So $\frac{d\sigma}{d\Omega}$ is constant. The total cross section is

$$\begin{aligned} \sigma &= \int \frac{d\sigma}{d\Omega}\, d\Omega \\ &= \frac{a^2}{4}\int_0^{\pi} 2\pi\ \sin\theta\, d\theta = \pi a^2 \end{aligned}$$

This is the cross-sectional area of the sphere, which is what we should expect.